Indian Statistical Institute, Bangalore Centre.
Supplementary Exam : Probability 1

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Max. points : 50. **Time Limit : 3 hours.**
Answer all five questions. All questions carry 10 points.

Give necessary justifications and explanations for all your arguments. If you are citing results from the class, mention it clearly. Always define the underlying random variables and events clearly before computing anything !

1. Let $\alpha > 0$ and X be a random variable with the pdf given by
$$f(x) = \frac{\alpha}{x^{\alpha+1}}, 1 \leq x < \infty \ ; \ f(x) = 0, \ x < 1.$$
   (a) Find the distribution of the following random variables $X_1 = X^2, X_2 = X^{-1}$. (**3**)
   (b) Find $k \in \mathbb{N}$ such that $\mathbb{E}\big[X_1^k\big]$ is finite and compute $\mathbb{E}\big[X_1^k\big]$ in such cases. (**4**)
   (c) Find $k \in \mathbb{N}$ such that $\mathbb{E}\big[X_2^k\big]$ is finite and compute $\mathbb{E}\big[X_2^k\big]$ in such cases. (**3**)

2. Suppose r distinguishable balls are arranged at random in $n(\geq 2)$ boxes. For each i, let us denote by X_i the number of balls in the ith box. Fix $m \in \mathbb{N}$ with $m \leq n$. Calculate the expectation of the following random variable
$$\frac{\sum_{1 \leq i < j \leq m} X_i X_j}{\sum_{1 \leq i < j \leq n} X_i X_j}.$$

3. Let $X_1, \ldots, X_n$ be i.i.d. Geometric random variables with parameters $p_1, \ldots, p_k$ respectively where $p_1, \ldots, p_k > 0$.
   - Find the pmf of $X = \min\{X_1, \ldots, X_n\}$ and identify the random variable as well. (**5**).
   - Let $p_1 = p_2 = p > 0$. Find the conditional pmf $p_{X_1|X_1+X_2}(.|.)$. (**5**).

4. Let $X_1, X_2, \ldots,$ be i.i.d. random variables such that $\mathbb{E}[X_1] = 0, \mathsf{VAR}(X_1) < \infty$. For every $n \geq 1$, let $Y_{i,n}, 1 \leq i \leq n$ be independent Geom$(\frac{i}{n})$, $1 \leq i \leq n$ random variables and also independent of $X_1, X_2, \ldots,$. Let $S_n = \sum_{i=1}^{n}(X_i + Y_{i,n})$. Show that for all $c > 0$,
$$\lim_{n \to \infty} \mathbb{P}(S_n \geq n \log n + cn) \leq \frac{\pi^2}{(c-1)^2 6}.$$

Hint : You can use that $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$ *and for all* $n \geq 1$, $|\sum_{i=1}^{n} \frac{1}{i} - \log n| \leq 1$.

5. Let there be N questions in an exam. We assume that N is a random variable with Poisson(λ) distribution. A student answers questions with probability p (for some $p \in [0,1]$) and a student answering a question is independent of answering other questions. Let Q denote the number of questions answered by the student and let $W = N - Q$. Compute the pmfs of Q, W and joint pmf of the vector (Q, W). Are Q and W independent random variables ?